The ppd, and the gun.

Problem.

Given a parallelepiped of fixed surface area and variable volume, prove that the maximum volume occurs when the ppd. is a cube.

Solution.

It's not difficult to prove that the local maximum must occur and must occur at the boundary because the only critical point of $V(a,b,c) = abc$ is $(0,0,0)$. That's not the sexy.

Let a be the height of one of the parallelogram sides. Let b, c be the other sides such that $V = abc$. Then the surface area $S = 2(ab + bc + ac)$, implying

$$a = \frac{S - 2bc}{2b + 2c} \tag{1}$$

$$V = \frac{Sbc - 2b^2c^2}{2b + 2c}. \tag{2}$$

Let's find the critical points of this. Partial derivatives:

$$V_b = \frac{(2b+2c)(Sc - 4bc^2) - (Sbc - 2b^2c^2)(2)}{2b+2c}$$

$$V_c = \frac{(2b+2c)(Sb - 4b^2c) - (Sbc - 2b^2c^2)(2)}{2b+2c}.$$

Setting them to zero, the most dangerous of all numbers:

$$\begin{aligned} 0 = V_b &= 2Sbc + 2Sc^2 - 8b^2c^2 - 8bc^3 \\ &\quad - (Sbc - 2b^2c^2)(2) \\ 0 = V_c &= 2Sb^2 + 2Sbc - 8b^3c - 8b^2c^2 \\ &\quad - (Sbc - 2b^2c^2)(2). \end{aligned}$$

Due to $V_b = 0 = V_c$, this implies

$$\begin{gathered} 2Sc^2 - 8bc^3 = 2Sb^2 - 8b^3c \\ b = c. \end{gathered}$$

(N.B. Another solution to this system is $S = 4b^2$. This is obviously a dead end. Why?)

What now? We need to prove $a = b$ somehow. This step took me two days, altogether 4 hours of thinking, to find. You plug it back into not Equation (1) but V_b, V_c. And carefully.

$$\begin{aligned}
0 &= V_b \\
&= 4b(Sb - 4b^3) - (Sb^2 - 2b^4)(2) \\
&= 4b^2(S - 4b^2) - 2b^2(S - 2b^2) \\
&= 2b^2(2S - 8b^2 - S + 2b^2) \\
&= 2b^2(S - 6b^2) \\
b &= \sqrt{S/6}.
\end{aligned}$$

Now,

$$\begin{aligned}
a &= \frac{S - 2bc}{2b + 2c} \\
a &= \frac{S - 2(S/6)}{4\sqrt{S/6}} \\
a &= \frac{2S/3}{4\sqrt{S/6}} = \frac{S/3}{2\sqrt{S/6}} \\
a &= \sqrt{S/6} = b.
\end{aligned}$$

$a = b = c$, whence a cube.

About the author

This simple problem was posted on what many people call the "Internet" first. Hao Lian publishes *The Dark Balloon*. He once ate an orange without making a mess.